Corrections to Test

1. On a separate piece of paper, redo any questions where you missed points.
2. Explain math mistakes made in the original solution.
3. If 1 & 2 are correct, then you will get $\frac{1}{2}$ the points back on that question.
4. You can redo this as many times as needed before Nov 2.

Subsequences

If $(a_n)_{n\geq 1}$ is a sequence and $1 \leq n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of positive integers, then $(a_{n_k})_{k\geq 1}$ is a called a <u>subsequence</u> of (a_n).

Examples: ① $\left(1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$ is a subsequence of $\left(\frac{1}{n}\right)_{n\geq 1}$.

② $(1, 1, \dots)$ is a subsequence of $((-1)^n)_{n\geq 0}$.

Notice the original sequence diverges, but the subsequence converges.

Lemma (Subsequences converge to the same limit)

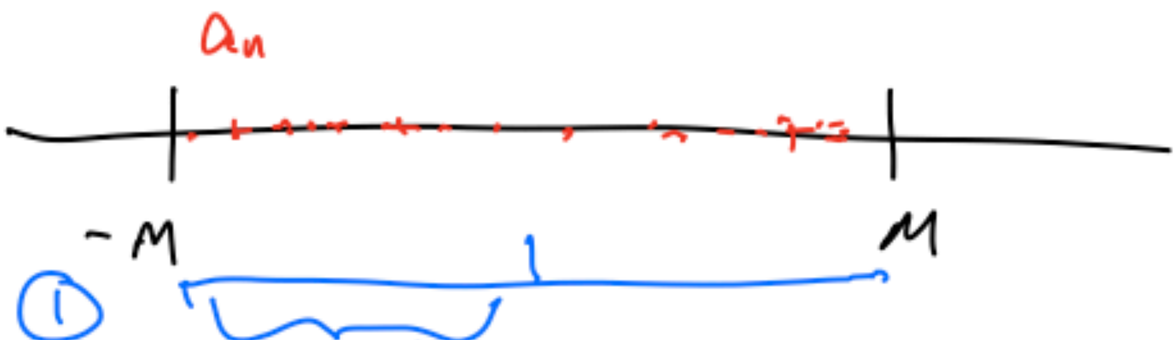
If (a_n) is a convergent sequence with $\lim a_n = L$, and if $(a_{n_k})_{k\geq 1}$ is a subsequence of a_n, then it converges to the same limit, i.e. $\lim\limits_{k\to\infty} a_{n_k} = L$.

Pf. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$. Observe that $\forall k \in \mathbb{N}$, $n_k \geq k$. Thus $\forall k \geq N$, then $n_k \geq N$, so $|a_{n_k} - L| < \varepsilon$. Thus, $\lim\limits_{k\to\infty} a_{n_k} = L$. ∎

Bolzano-Weierstrass Theorem

Every bounded sequence contains a subsequence that converges.

Idea of Proof - Say (a_n) is bounded, i.e. $\exists M > 0$ s.t. $-M \le a_n \le M$ $\forall n \in \mathbb{N}$.



① pick side that contains an ∞ # of elements of the sequence.

Pick a_{n_1} in here

② pick side that contains a ∞ # of elements of (a_n) with $n > n_1$

choose a_{n_2} from those elmts.

③

pick side

$\vdots$

$a_{n_3} \cdots$

$\ddots$

The intervals converge to a point L, and the constructed $(a_{n_k}) \to L$.

"$\blacksquare$"

Lim sup & Lim inf.

Exercise 2.4.7 (Limit Superior). Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

ⓐ Prove that $\left(y_n = \sup\{a_k : k \geq n\}\right)$ converges.

(We call the limit $\limsup a_k$.)

Scratchwork — (a_k) bounded.

(y_n) is not a subseq. of (a_n)?

$$y_{n+1} = \sup\{a_k : k \geq n+1\}$$
$$\leq \sup\{a_k : k \geq n\} = y_n$$

$(1, 2, 3, 4, 1, 5, 1, 1, \ldots)$

(y_n) is a decreasing sequence.

$y_1 = 5$

$y_2 = 5$

$y_3 = 5$

$y_4 = 5$

$y_5 = 5$

$y_6 = 5$

$y_7 = 1$

$-M \leq a_n \leq M \quad \forall n \in \mathbb{N}.$

$-M \leq a_n \leq y_n \leq M$

because M is an upper bound for $\{a_k : k \geq n\}$.

$\therefore$ (y_n) is bounded.

$\therefore$ by MCT (y_n) converges. $\boxtimes$

(b) $\liminf a_n = \lim_{n\to\infty} z_n$,

where $z_n = \inf\{a_k : k \geq n\}$.

(c) Prove $\liminf(a_n) \leq \limsup(a_n)$

Give example where it is $<$.

Pf.

$\liminf(a_n)$

$= \lim \inf\{a_k : k \geq n\}$.

and $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$

By OLT,

$$\liminf a_n \leq \limsup a_n. \quad \boxed{*}$$

Example where $\liminf a_n < \limsup a_n$

Let $a_n = (-1)^{n+1}$

$$\liminf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$$

$$= \lim_{n\to\infty} (-1) = -1$$

Simi $\limsup a_n = 1$.

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(2) Thm For any bounded sequence a_n, $\lim a_n$ exists

$\iff \liminf a_n = \limsup(a_n)$.

(And all 3 limits are the same).

Pf. ($\Longrightarrow$)

If $\lim a_n = L$, then the subsequence $(a_n, a_{n+1}, \cdots)$ also has limit L.

By defn of limit, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|a_n - L| < \varepsilon$$

$$\Longleftrightarrow \quad L - \varepsilon < a_n < L + \varepsilon.$$

for $n \geq N$.

Thus
$$\sup_{n \geq N} a_n \leq L + \varepsilon$$

because $L + \varepsilon$ is an upper bound for $\{a_k : k \geq N\}$.

Similarly, $L-\varepsilon \leq \inf_{n \geq N} a_n$

because $L-\varepsilon$ is a lower bound for $\{a_k : k \geq N\}$.

Thus,

$$L - \varepsilon \leq \inf \{a_k : k \geq N\}$$
$$\leq \sup \{a_k : k \geq N\} \leq L+\varepsilon.$$

Also, it is true for N replaced by any $M \geq N$.

$\therefore \forall M \geq N$

$$L - \varepsilon \leq \inf \{a_k : k \geq M\}$$
$$\leq \sup \{a_k : k \geq M\} \leq L+\varepsilon.$$

Taking the $\lim_{M \to \infty}$ & using OLT

$$\Rightarrow L-\varepsilon \leq \liminf a_n$$
$$\leq \limsup a_n \leq L+\varepsilon.$$

This inequality is true $\forall \varepsilon > 0$, so if we let $\varepsilon = \frac{1}{p}$, for $p \in \mathbb{N}$:

$$L - \frac{1}{p} \leq \liminf a_n \leq \limsup a_n \leq L + \frac{1}{p}.$$

Use OLT, $\lim\limits_{p \to \infty}$ to get

$$L \leq \liminf a_n \leq \limsup a_n \leq L.$$

$$\therefore \quad \liminf a_n = \limsup a_n = L. \ \checkmark$$

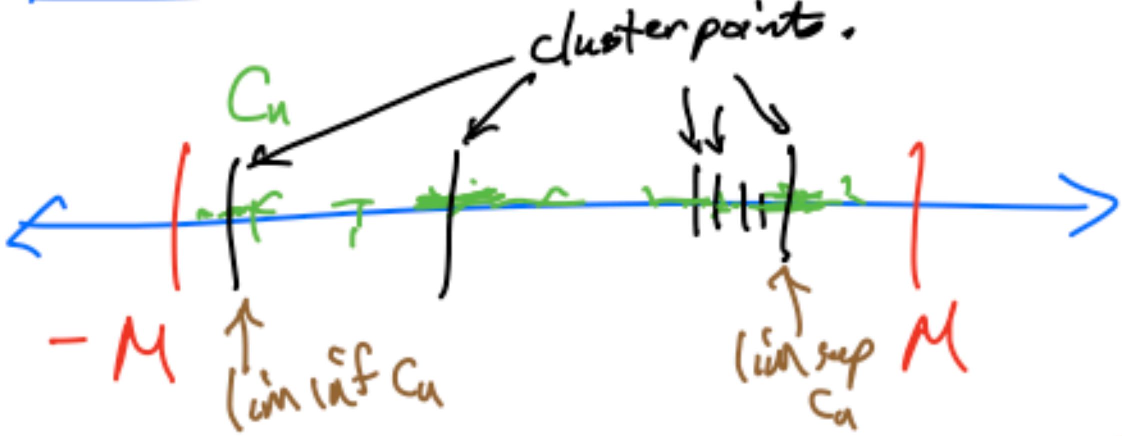
Assume we proved the other direction

More information about bounded sequences.

Given a bounded sequence (c_n) we say a number x is a <u>cluster point</u> (or <u>limit point</u>) of (c_n) if $\forall \varepsilon > 0$, $\exists$ subsequence (c_{n_k}) of (c_n)

such that $x - \varepsilon < c_{n_k} < x + \varepsilon$ $\forall k \in \mathbb{N}$.

Thm — For any bounded sequence (c_n), $\limsup c_n$ = greatest cluster point of (c_n), and $\liminf c_n$ = least cluster point of (c_n).



Example. Let $(q_1, q_2, \ldots)$ be a sequence of all the rational numbers in $(0,1)$. Then every $x \in [0,1]$ is a cluster point of (q_n).

(using fact that $\mathbb{Q}$ is countable)